An Analysis of the Hidden Structure Behind the Chaos of the Williamowski-Rössler Network

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We adore chaos because
we love to produce order.
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Finally, my thanks and praise go to God, who created this beautiful, crazy, and chaotic world for us to explore.

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Introduction

The history of humanity's love affair with knowledge consists of finding reason and rhyme to the seeming madness of nature. Science has been at the head of this charge, attempting to light the dark room in which we have found ourselves. Science's light of explanation is a confident one which asserts that given one scenario another scenario will inevitably follow. This approach is startlingly fatalistic, and this fatalism has especially taken hold of the 20th and 21st centuries, spawning philosophies such as naturalism and its blunter cousin, nihilism. The discovery of chaotic mechanisms both affirms and laughs at this confidence. On the one hand, it confirms that structures within nature are determined, given a certain set of parameters, one result inevitably follows. On the other, it asserts that there is no possible way for us to know the result. This thesis will hopefully contribute to an assault of this latter proposition, in discussing the hidden structure behind the chaotic behavior of the Williamowski-Rössler Network.

First, we should define our terms. Plato wisely suggests, "For if your starting-point is unknown, and your end-point and intermediate stages are woven together out of unknown material, there may be coherence, but knowledge is completely out of the question."[i] So what is chaos? Steven H. Strogatz proposes the following as a working definition, noting that there has yet to be a universally accepted definition of such an esoteric idea, "***Chaos*** is *aperiodic long-term behavior* in a *deterministic* system that exhibits *sensitive dependence on initial conditions*."[ii] We can define rules for chaos then:

(1) It is *aperiodic*, it will not converge to any point or periodic pattern, though it is certainly not random, leading to our second requirement.
(2) It is *deterministic*, there is only one result for one set of parameters.
(3) It displays *sensitive dependence on initial conditions*, a small perturbation between two sets of parameters will diverge exponentially fast.

A frequent analogy to this structure is called "the butterfly effect." This term, coined by Edward Lorenz, was first discussed by Ray Bradbury in the short story, *A Sound of Thunder*. In this story, a group of time travelers go back in time, and one of them inadvertently crushes a butterfly beneath his boot. They go back to the future to find, much to their chagrin, that the world has been dramatically altered in a way such that they could never have possibly foreseen. The death of the butterfly was a small alteration in the set of parameters of the universe, but the trajectory of the future was altered in a way that was inherently unpredictable. Note, it was still determined; this change to the future was *because* of the butterfly's untimely death. And the future, being the strange thing it is, does not converge to a point or pattern.[†] So this example is an excellent glimpse into a chaotic universe.

Chaos, it appears, is much easier to describe by what it is not. Any system that converges to equilibrium or periodicity is not chaotic. Any system that is random and therefore not determined is not chaotic. And any system that does not have sensitivity to its initial conditions is not chaotic. So a chemical reaction that converges to equilibrium is not chaotic, nor is the system that oscillates regularly between some set of finite points.

The first two requirements are fairly easy to understand. Determinism and aperiodicity are intuitive, what of sensitivity? Sensitivity is a condition which prevents predictability based on similar sets of parameters. If a system is not sensitive, it will not be too difficult to predict the behavior if one changes the parameters ever so slightly. Sensitivity makes this impossible; the slightest change in any parameter in a chaotic system will ultimately result in a drastically new result. We can measure this by observing the difference of populations in a system over n

[†] This certainly is a more philosophical claim, and the weakest made in this analogy, but the point remains.

iterations over two systems that are identical with the exception of one small perturbation. We can set this up by setting the differences equal to each other, but pairing one with an exponential:

$$\delta_n = \delta_0 e^{\lambda n}$$

In this definition, n is number of iterations, δ_0 is the perturbation, the initial difference, and δ_n is the difference in conditions after n iterations. λ is known as the Liapunov constant. Some algebraic trickery will result in:

$$\lambda = \frac{1}{n}\ln\left(\frac{\delta_n}{\delta_0}\right)$$

If λ is positive, then the system is sensitive, implying that it is a chaotic system; slight changes in initial conditions will result in an exponential difference in results.[iii]

Note that the combination of the first and third rule (see page 1) renders any chaotic system incredibly difficult to predict. There is no convergence to a point or periodic behavior. There is no way of telling how one set of parameters will behave based on the behavior of other sets of parameters. Yet the second rule keeps us from saying that it is random.

So, this is what chaos is. But how do we map chaos? What does it look like? And most importantly, is there a method to the madness behind chaos? The purposes of this thesis will be first to highlight some different ways to view chaos, and then looking into the underlying structure. This thesis will be building off of the work of the senior thesis of Aaron Bush[iv], whose examinations into the structure of chaos have been nothing but illuminating and insightful.

Phase Portraits

One basic way to qualitatively examine chaos is found in mapping out its attractor, constructed by plotting the present vs. the future. This is known as a phase portrait. The horizontal axis of such a graph is the system at time t, plotted against the vertical axis of the graph, the system at time $t + \tau$, where τ is a constant time step size. For the following section we set $\tau = 300$. This results in a structure that clearly indicates the where a system goes, indicating convergence, periodicity, or a lack of either.

The phase portrait of a convergent system will eventually come to rest at a single point, making no movement after this. This is due to the fact that the value of the system t and $t + \tau$ approach the same result, so eventually, as $t \to \infty$, $f(t) = f(t + \tau)$. Displaying this is Figure 1, as system that is convergent. Accordingly, its phase portrait converges to equilibrium. Notice that our portrait's path begins roughly at (.2, .62) and after a short stint of oscillations, settles at roughly (1.2,1.2). It will not move after this, for $f(t) = f(t + \tau)$ as $t \to \infty$.

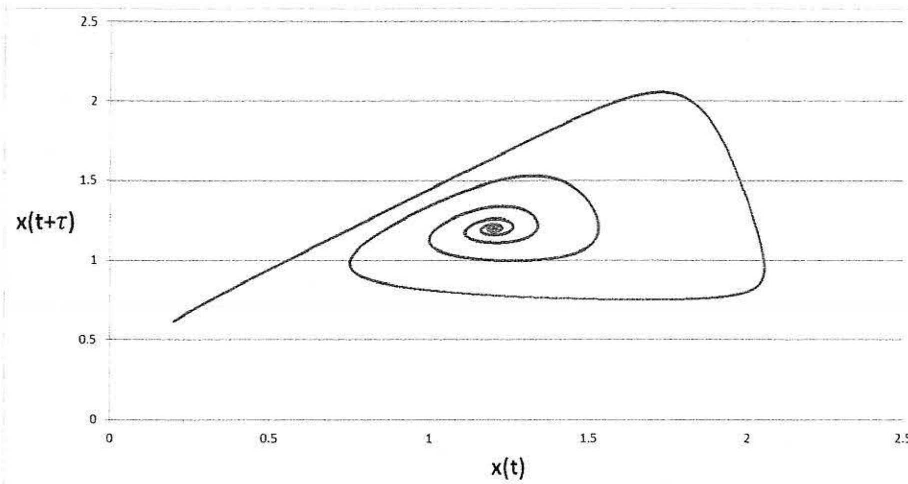

Figure 1: A convergent system approaching a single point.

A periodic system will be modeled as a loop, given that, for every value x_1 at time t_1, there is a $t_2 > t_1$ such that the value x_2 at t_2 will equal x_1. In essence, there is only a limited amount of values in

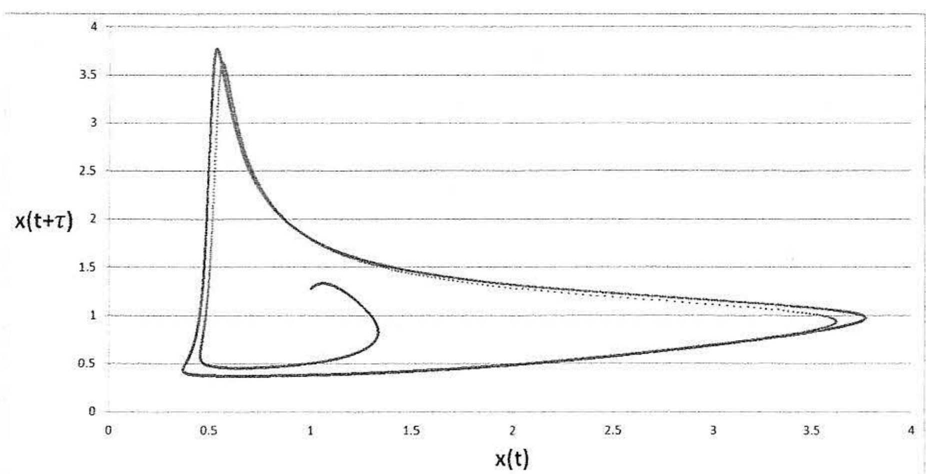

Figure 2: An system approaching stable periodicity.

the system, and so it will constantly loop back on itself. The phase portrait of a system that settles to periodicity (Figure 2) displays this behavior, starting at approx. (1, 1.277) and quickly settling into an infinite loop, or limit cycle.

A random, noisy system will look just that in the phase portrait (Figure 3). Given that there is no rhyme or reason to the system, no consistent pattern, there will be no correlation between t and $t + \tau$. So the points on the phase portrait will be as random and noisy as the system they model.

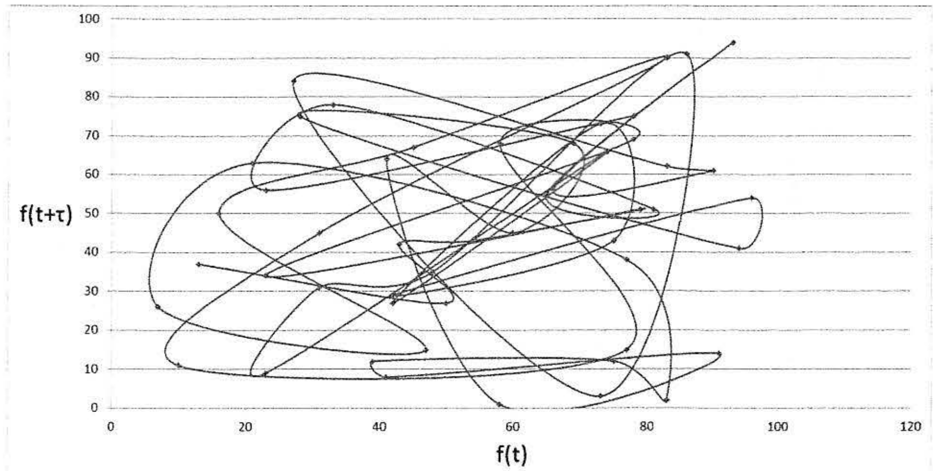

Figure 3: A random, noisy system.

What about chaotic systems? These systems are not noisy (determined), but they are not convergent to periodicity or to a point. They are *aperiodic*, so they will not be repeating themselves regularly; however, given that it continually goes through a similar pattern, the phase portrait will continually loop back on itself in varying paths. Consider the Williamowski-Rössler Network (Figure 4), which we will soon explore. It is clear that x has no periodicity, there is no infinite loop it settles into. However, it is certainly not random.

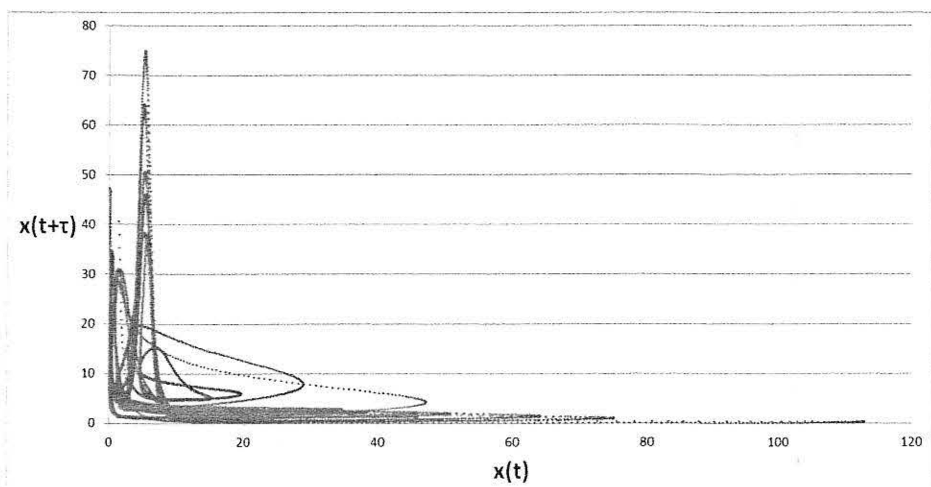

Figure 4: A chaotic system.

Bifurcation Diagrams

There are several ways to view chaos, phase portraits are certainly not the only way. One of the more powerful methods is the bifurcation diagram. The main contribution of Aaron's thesis lies in these diagrams, in which he was able to find underlying structure. This thesis will first explore bifurcation diagrams, explaining how they are constructed and what exactly it is they show. Code in Java has been written to analyze these diagrams for underlying structure, continuing Aaron's work and pushing it further.

A bifurcation diagram takes a system and, varying one parameter of the system, tracks the changes that follow from this variation. So instead of looking at progress over time, it looks at progress over the change of a certain parameter. While progress over time will result in a one-to-one mapping, progress over the change of a certain parameter does not. For some parameter value, all results from some time t_n to another time t_m are mapped. Steps t_0 through t_{n-1} are ignored, as these consist of an "incubation period." Adding these values would only catch the system when it is starting up, which distracts us from the information we want.

The easiest example of a bifurcation diagram is found by analyzing the logistic equation, which is used to model population growth. The equation is defined as follows:

$$\frac{dp}{dt} = rp(1 - p)$$

Here, t is time, p is population, and r is the growth rate parameter. We can approximate this using Euler's method, yielding the following one dimensional map:

$$p_{n+1} = rp_n(1 - p_n)$$

This allows us to programmatically examine the logistic equation. This form of an equation matches program language very well. We will first examine r values of 2.9, 3.0, 3.45, and 3.8 with a starting population of .1, and then look at the progression of r values, forming a bifurcation diagram.

For $r = 2.9$, the population very quickly converges to roughly .655 (Figure 1.1). For $r = 3.0$ (approx.), the population begins oscillating between two values, approximately .678 and .654. For $r = 3.45$ (approx.), the population begins oscillating between four values. And for $r = 3.8$ the population becomes erratic, never coming to any sort of oscillatory behavior. So we see that as r increases, the system becomes more and more frantic, oscillating more wildly. Eventually, after $r = 3.5699$ (approx.)[v], the system becomes chaotic, reaching aperiodicity.

The progress of this is clearly seen in the bifurcation diagram (Figure 5). As r increases,

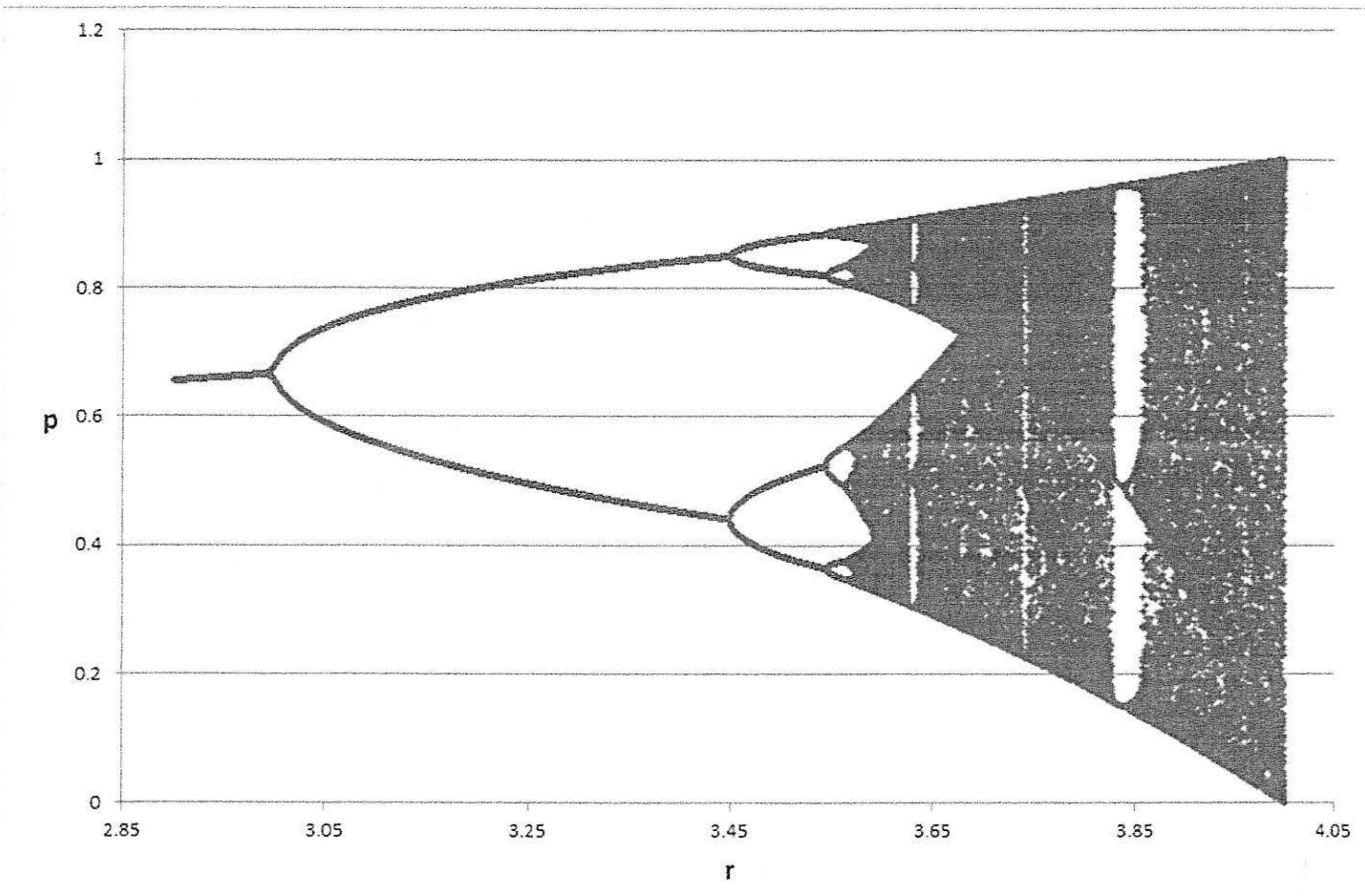

Figure 5: The bifurcation diagram of the logistic equation.

the population convergence becomes more frantic. And this gives us a visual representation of a transition from a stable system to a chaotic system. When r is 3.45, the logistic equation is not chaotic, but when r is greater than 3.5699 (approx.) we see aperiodicity completely take over.

The Williamowski-Rössler Network

It is not with the logistic equation that we are concerned with though. There is another system that displays chaotic behavior called the Williamowski-Rössler Network. This system is far more complicated than the logistic equation, being a multivariable system, all variables being dependent on another. The system is modeled by the following set of chemical equations:

Scheme 1

$$A_1 + X \underset{k_{-1}}{\overset{k_1}{\leftrightarrow}} 2X$$
$$X + Y \underset{k_{-2}}{\overset{k_2}{\leftrightarrow}} 2Y$$
$$A_5 + Y \underset{k_{-3}}{\overset{k_3}{\leftrightarrow}} A_2$$
$$X + Z \underset{k_{-4}}{\overset{k_4}{\leftrightarrow}} A_3$$
$$A_4 + Z \underset{k_{-5}}{\overset{k_5}{\leftrightarrow}} 2Z$$

Where $A_1, \ldots, A_5$ are constants, and X, Y, Z chemical variables. This translates into the following set of differential equations:

Scheme 2

$$\frac{dX}{dt} = k_1A_1X - k_{-1}X^2 - k_2XY + k_{-2}Y^2 - k_4XZ + k_{-4}A_3$$
$$\frac{dY}{dt} = k_2XY - k_{-2}Y^2 - k_3A_5Y + k_{-3}A_2$$
$$\frac{dZ}{dt} = k_5A_4Z - k_{-5}Z^2 - k_4XZ + k_{-4}A_3$$

For simplicity's sake we may absorb the A_ns into the k_ns, given that both are constants, resulting in:

$$\frac{dX}{dt} = k_1X - k_{-1}X^2 - k_2XY + 2k_{-2}Y^2 - k_4XZ + k_{-4}$$
$$\frac{dY}{dt} = k_2XY - k_{-2}Y^2 - k_3Y + k_{-3}$$
$$\frac{dZ}{dt} = k_5Z - k_{-5}Z^2 - k_4XZ + k_{-4}$$

Scheme 3

There are many ways to induce chaos, we will be looking into varying the k_3 variable from 9.85 to 10.0. We can create a bifurcation diagram with ease given the following one dimensional map by using Euler's approximation again:

Scheme 4

$$x_{n+1} = x_n + (k_1x - k_{-1}x^2 - k_2xy + k_{-2}y^2 - k_4xz + k_{-4})dt$$
$$y_{n+1} = y_n + (k_2xy - k_{-2}y^2 - k_3y + k_{-3})dt$$
$$z_{n+1} = z_n + (k_5z - k_{-5}z^2 - k_4xz + k_{-4})dt$$

Where dt is a very small step size. For the duration of this thesis, we will set $k_1 = 30, k_2 = 1,\ k_4 = 1, k_5 = 16.5, k_{-1} = .25, k_{-2} = .001, k_{-3} = .01, k_{-4} = .01$, and $k_{-5} = .5$, with our step size, dt as .0006. Recall that k_3 will be varied to induce chaos. Given this approximation set, it is very easily to programmatically construct both a phase portrait and a bifurcation diagram in the same way that we did with the logistic equation.

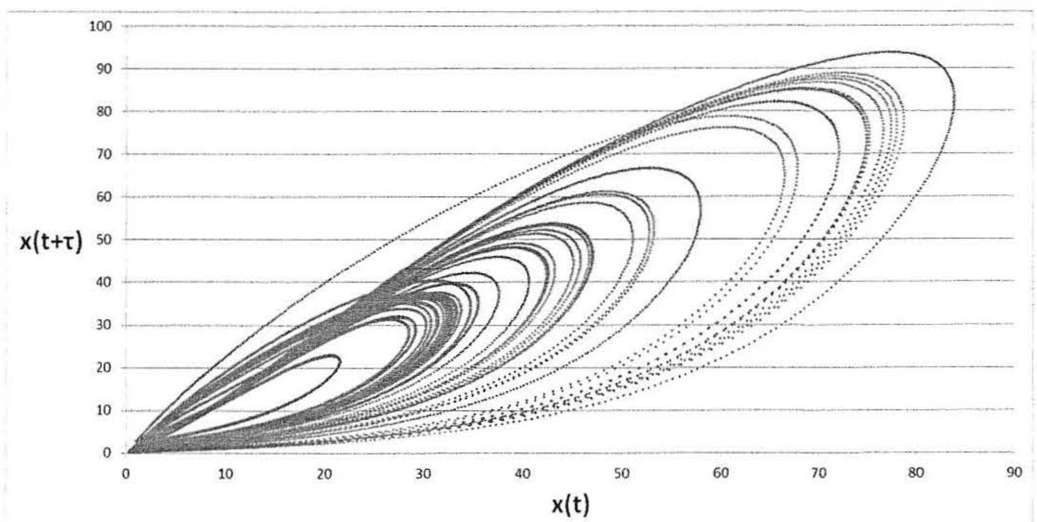

Figure 6: The phase portrait of the Williamowski-Rössler Network.

First though, we must confirm that it really is chaotic. We will run this system through the three tests prescribed by Strogatz, lack of convergence, determinism, and sensitivity.

To judge if there is convergence, or a lack thereof, we may use the phase portrait. Let us gain a glimpse into the behavior modeled by the phase portrait. We are tracking the behavior of x with $\tau = 60$ (Figure 6). Notice how the portrait is exactly how we would expect a chaotic system to be, with behavior that repeats in similar, but never the same, method. So we can rest assured that our system does not settle into any sort of periodic behavior, nor does it converge to a point. Our phase portrait clearly portrays this.

The second requirement is determinism, and this is passed without question. Clearly this system is determined, given a set of parameters and initial values, there can be only one outcome. Running a simulation with a set of parameters and initial values repeatedly yields the same result each time.

What of sensitivity? Consider our test for this, finding the Liapunov constant:

$$\lambda = \frac{1}{n}\ln(\frac{\delta_n}{\delta_0}).$$

We will first run the system with $x_0 = y_0 = z_0 = 1$, and then run it the same with the exception of $x_0 = 1.01$, so $\delta_0 = .01$. After 10,000 steps ($n = 10000$), the former's x value is 7.426937, the latter's is 11.15516. So $\delta_{10000} = 3.728222$.

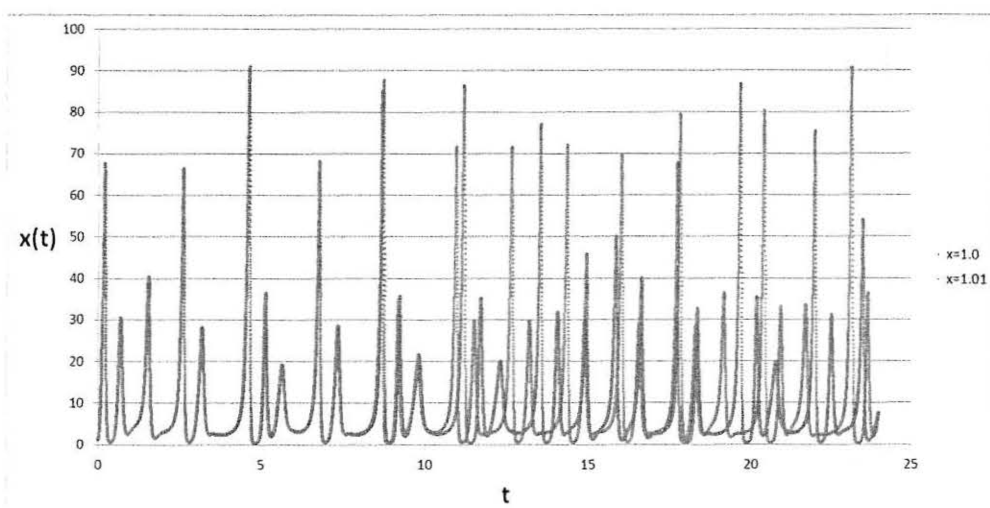

Figure 7: Comparing the networks with different starting x values.

Hence $\lambda_x = \frac{1}{10000}\ln\left(\frac{3.728222}{.01}\right) = \frac{1}{10000}\ln(372.8222) = \frac{1}{10000}(5.921102)$.

Therefore $\lambda_x = .000592$.

We may then run the system setting $y_0 = 1.01$, and again $z_0 = 1.01$ (Table 1). In changing y_0, we will guage λ by the corresponding differences in y values, and do the same with z. The respective λ values are $\lambda_y = .000726$ and $\lambda_z = .000661$. Given that λ is positive in all cases where $n = 10000$, we can safely claim that there is an exponential difference in the results given by two different initial starting values. In fact, even if we expand the test to include all variables for all modifications, the λ value is positive in all the test cases. Therefore we may conclude that this system passes the test of sensitivity.

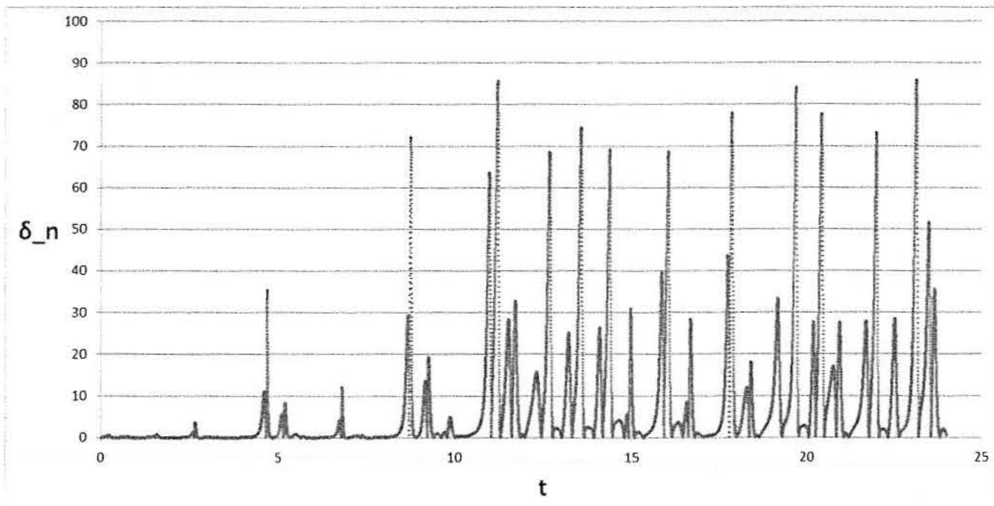

Figure 8: Comparing the differences in the divergent systems over time.

But how long does it take to display sensitivity? First, we gain an intuitive look at sensitivity in Figure 7, which plots the first 40,000 steps, each discrete step representing .0006 seconds. Notice how at around second 10 the two systems begin to widely differ, only having the same x value out of necessity, when one is rising and the other falling or some other coincidence.

Plotting the δ_n value against time (Figure 8), yields a look into the sensitivity of the system, and we can clearly see that it takes roughly 5 seconds for the system to become significantly different. This matches perfectly with Figure 7, as the first divergence is at the 5th second.

Plotting the Liapunov constant against step yields a simlar result. Note that a system need not have a negative λ value at all times for it to be chaotic. Indeed, this would be impossible, as there are only a finite number of values that a system may have. A negative λ value indicates a small difference between the two systems at that point in time, which may be explained by the fact that the two systems simply happened to be at a similar or same value x for some step n. Given that the system is contained within a finite set of points and is not convergent, this is unsurprising.

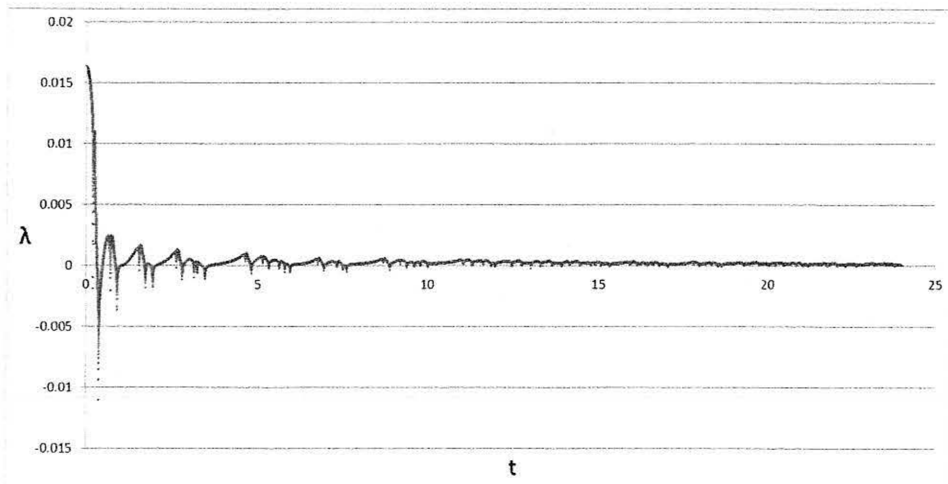

Figure 9: The Liapunov constant of the two divergent systems over time.

All three tests have been passed, the Williamowski-Rössler is chaotic for our set of parameters. But what does this chaos look like, how can we describe it? We have a sample of it with our phase portrait, can we go further?

λ values	$x = 1.01$	$y = 1.01$	$z = 1.01$
λ_x	$.000592$	$.00051$	$.000455$
λ_y	$.000576$	$.000726$	$.000655$
λ_z	$.000577$	$.00073$	$.000661$

Table 1

And it is not with the phase portrait that our main focus resides. It is in the more telling bifurcation diagram that we are most interested. However, upon creation of the diagram we find that it does not share some of the distinctive elements of the logistic equation's (Figure 10). This is due to the fact that there is no convergence to a single point in this network, so this bifurcation diagram rather shows the range of possible values for each k_3 value.

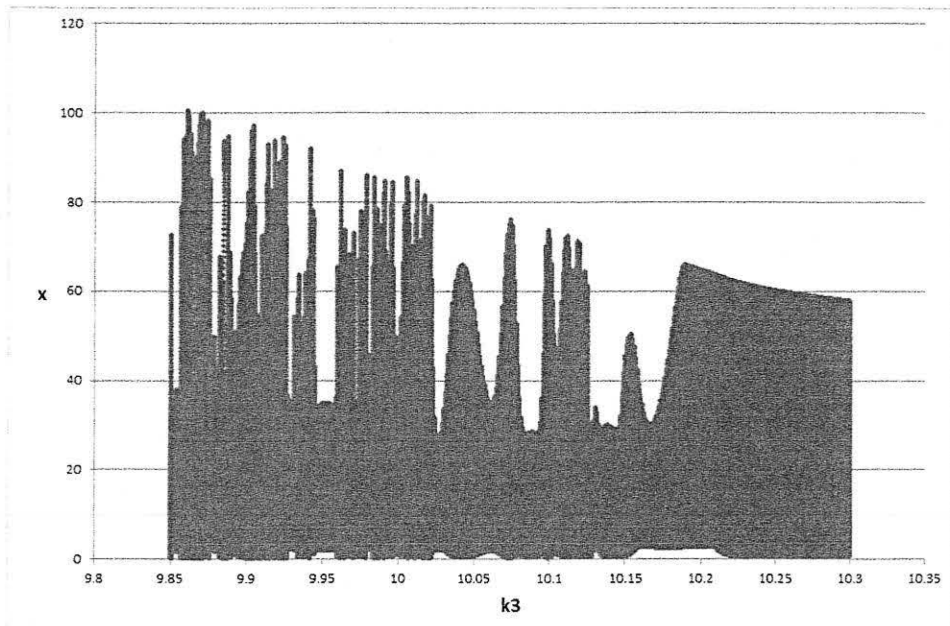

Figure 10: The bifurcation diagram of the Williamowski-Rössler Network.

This is certainly not very useful, however, all is not lost. We can create a useful bifurcation diagram by only mapping out the peaks and troughs of the network. While a tedious and difficult task by hand, this is easily accomplished programmatically. While Aaron's program to accomplish this involved approximating a derivative to discover if there was a peak, the program constructed for this thesis simply checked $if(x_{n-1} < x_n > x_{n+1})$ for a peak at x_n and $if(x_{n-1} > x_n < x_{n+1})$ for a trough at x_n. If the previous and the next x value are less than the current, then clearly the x is a peak. The same logic holds for the trough test. The bifurcation diagram created by this technique has the characteristic patterns found in the logistic bifurcation diagram (Figure 11).

So instead of marking where the system itself converges, the peak and trough amplitudes will be observed (Figures 11-13). With this system, interestingly enough, we find that as k_3 increases, the peak and valley amplitudes converge to oscillation between four points, unlike the logistic equation, which itself branched out rather than losing branches (Figures 11-13, $k_3 > 10.2$).

We can go further though. This only modeled the progress of the x variable, we can model y (Figure 12) and z (Figure 13) as well, and then proceed to investigate any hidden patterns that the resulting graphs may have buried beneath them. Notice that the structures are very similar. Also notice the subtle wave patterns inherent in them. It is in these patterns that we are primarily concerned, and that the majority of this thesis will investigate.

So, it took stripping away all values other than peaks and troughs to acquire some sense of order from the bifurcation diagram. The peaks and troughs therefore are of significance, and these modified bifurcation diagrams show us this very clearly.

Our next step will therefore be to track specific peaks. We will continue Aaron's work by investigating the 15th peak of this network, and see what conclusions we may draw.

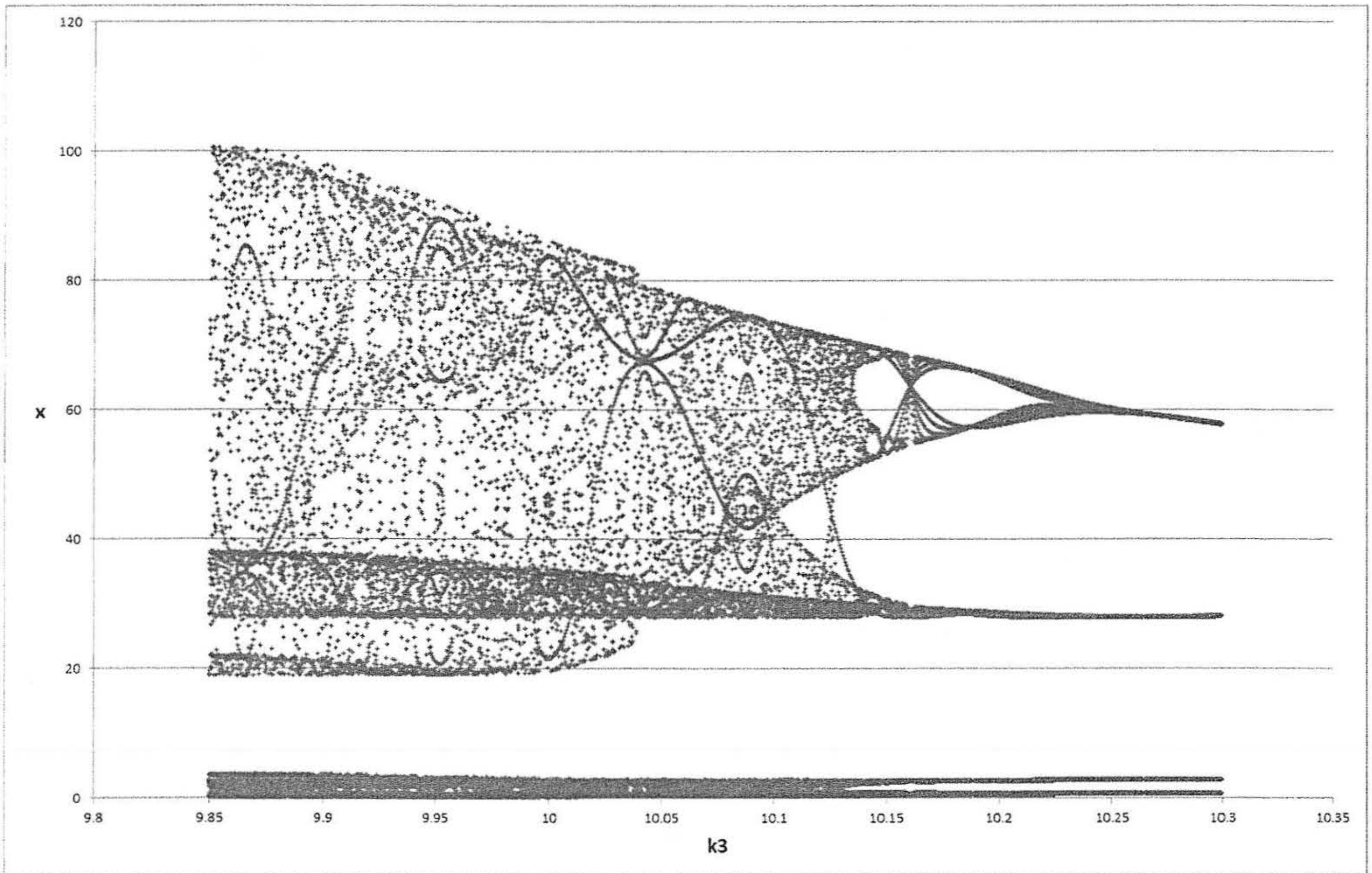

Figure 11: The bifurcation diagram of the Williamowski-Rössler Network, only considering x peak and valley values.

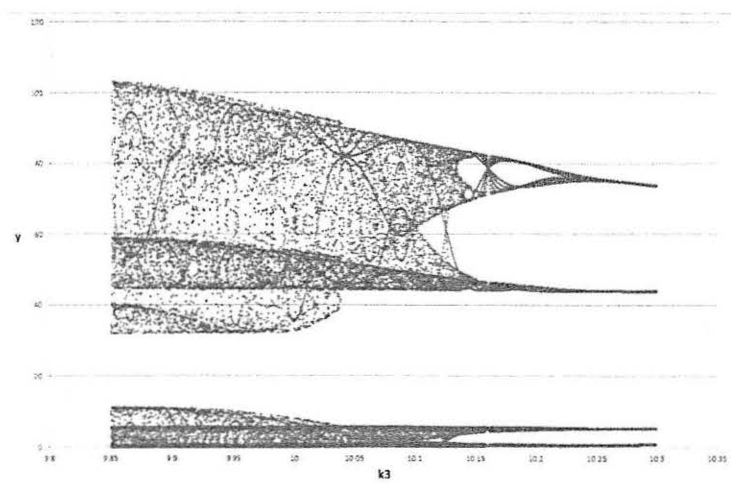

Figure 12: WR network bifurcation diagram, y peak and valley values

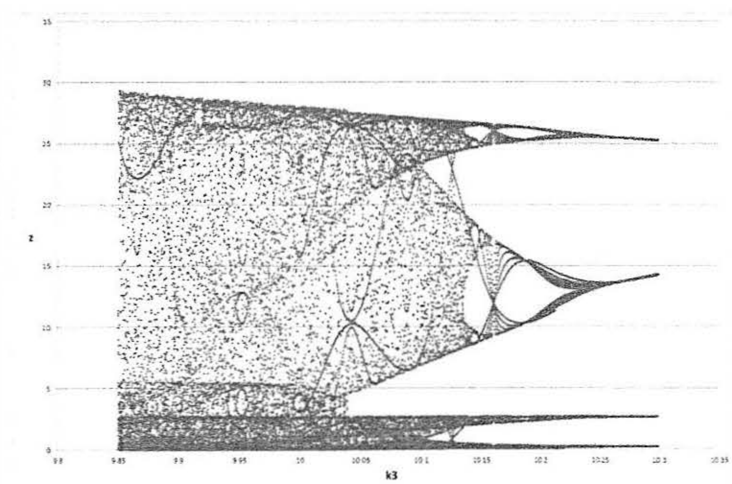

Figure 13: WR network bifurcation diagram, y peak and valley values

Investigating the 15th Peak

Programmatically, it is not a difficult task to pull out the nth peak of any set of equations, and the Williamowski-Rössler Network is no exception. By tracking the 15th x peak, we obtain the result shown in Figure 14, with a higher resolution shown in Figure 15. Notice that there are sections of continuity, however, notice also that these sections are choppy, going from sections of continuity to sections of continuity very abruptly. While there are hints of a waveform, it is a very broken waveform.

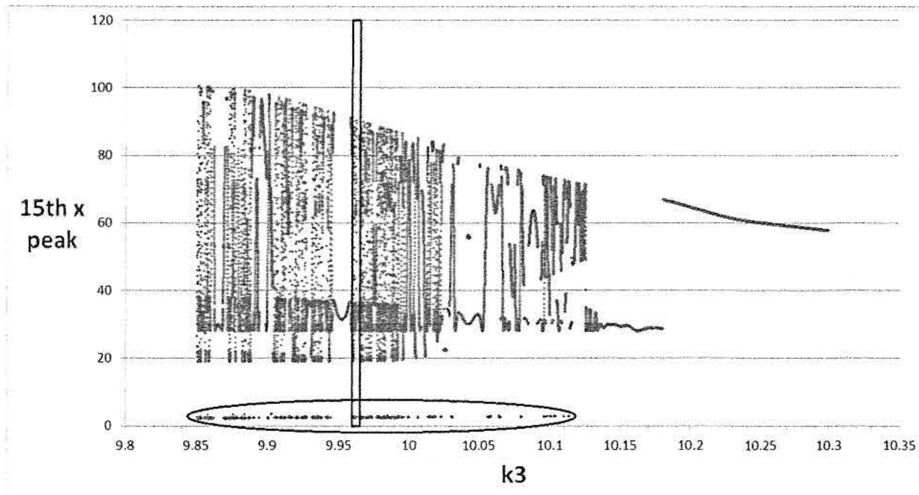

Figure 14: The 15th x peak values for each k_3 value.

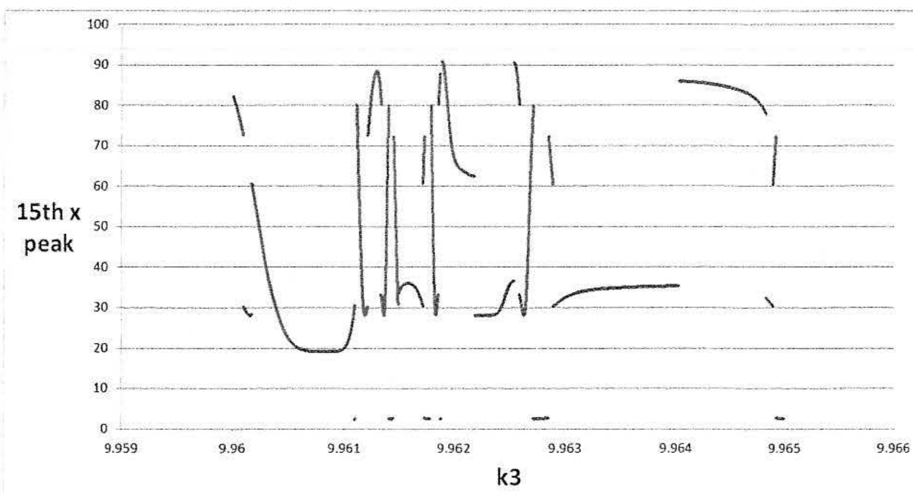

Figure 15: The 15th x peak values for each k_3 value, with higher resolution.

Even sections of continuity is better than none at all though, and there is a way that we can force some pattern to emerge from this, much like we drew a pattern (Figure 11) out of our original bifurcation diagram (Figure 10). One element of this peak graph is the steady amount of very low peaks (Figure 14, circled data points). What if we could remove these holes? Would the graph fill itself in? Surprisingly, yes, it does just that (Figure 16).

The holes are removed by enforcing a minimum. We do so by requiring x to be greater than ten in order to qualify as a peak, 15th peak or not. We will call this practice *inflating*, and any analysis on a system using this technique will be called an *inflated look*. The minimum value will be ω. By artificially inflating the peaks in such a way, requiring any peak value x to be greater than $\omega = 10$, we find, oddly enough, a much more consistent continuity (Figure 16). Increasing the resolution to examine some of the questionable areas reveals continuity as well (Figure 17), for example, from 9.96 to 9.965.

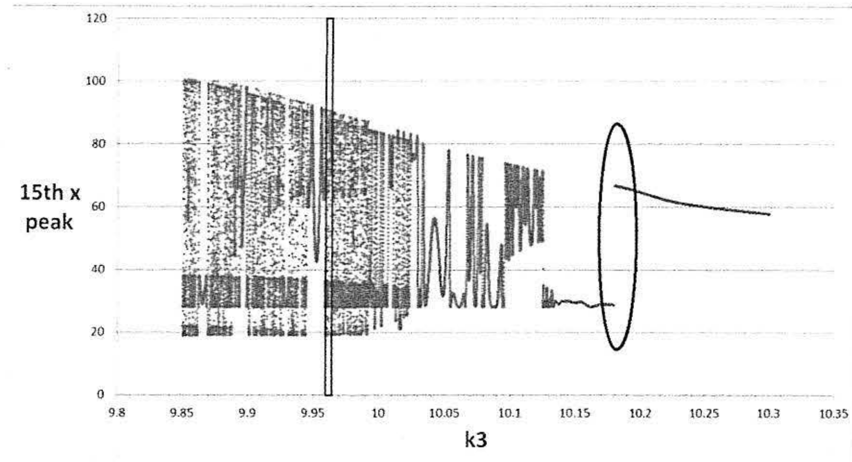

Figure 16: The 15th x peak values for each k_3 value, $\omega = 10$.

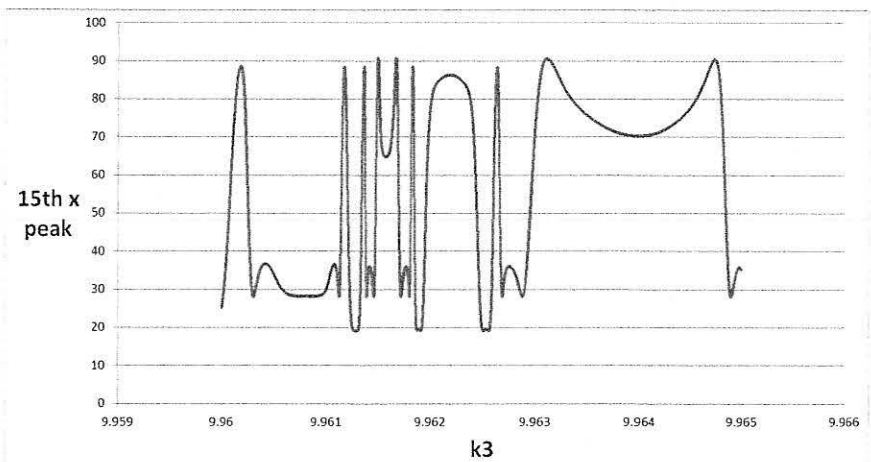

Figure 17: The 15th x peak values for each k_3 value, $\omega = 10$, with higher resolution.

The very low peaks in the uninflated peak tracker disappear with inflation, and continuity results. This is surprising. We have not taken these low values and added 10 to them or any such thing. We have entirely replaced them with new ones. Consider that there exists some 13th peak that is below ten. Since this does not qualify for our inflated analysis, the 14th peak would take its place, if it is above 10. This process therefore shifts the peaks around in ways that are difficult to predict. This is why the graphs of the inflated and uninflated systems do not match (Figures 14 and 16, 15 and 17). It is not as if we only removed the lower peaks and that is all. Rather, we have uncovered an entirely new set that is continuous. The 15[th] x peaks that are above $\omega = 10$ are continuous.

There is, however, one notable and obvious exception. Figure 16, our inflated look at the system, which is continuous everywhere except between 10.1801215450983 and 10.1801215450984 (Figure 16, circled discontinuity). Here there exists a leap from approx. 28.73441 to 66.84738. This is quite anomalous, very much out of sync with the rest of the graph. I leave as future study the question of why this is.

This anomaly aside, we have continuity on the 15[th] peak of x (Figure 16, 17), but we only have it by artificially inflating the peaks, discounting any peaks that are below 10.

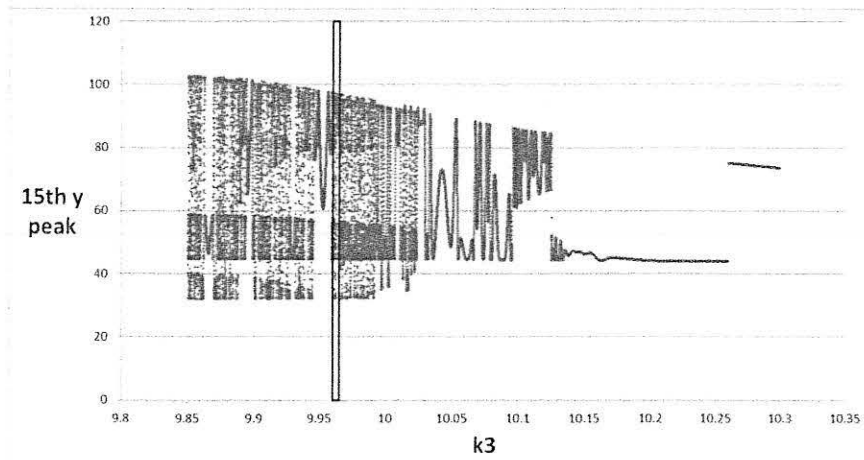

Figure 18: The 15[th] y peak values for each k_3 value.

Let us consider the other values of the system, y and z. We may track these in the same way that we tracked the peaks of x. Given that we noticed similar structure in the bifurcation diagrams of y and z, we should have very similar

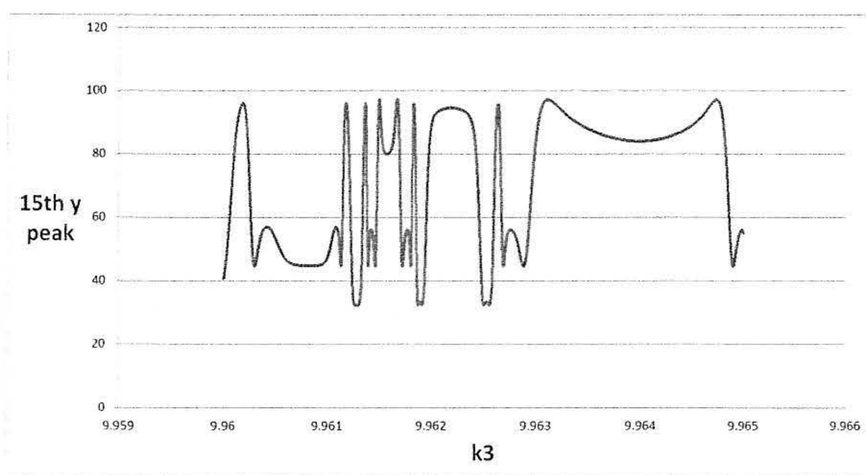

Figure 19: The 15[th] y peak values for each k_3 value, with higher resolution.

results. So, tracking the 15th peak of y and z from 9.85 to 10.3 will yield Figures 18 and 19 for y, as well as 20 and 21 for z.

Notice how the 15th peak for y (Figures 18-19) and z (Figures 20-21) are both continuous throughout. Increasing the resolution only confirms this (Figures 19, 21). However, unlike the graph of x's 15th peaks, these are continuous despite being uninflated. y and z both have similar anomalous breaks, but at different points than x, which is certainly an oddity. Upon closer inspection, it is revealed though it appears that z has two breaks, it is really only one. An increase in resolution shows that the only disconnect is at slightly above 10.11, the apparent disconnect at approx. 10.125 is actually continuous. But that aside, the fact remains that, with all peaks allowed to be whatever size, y and z's peaks remain continuous throughout.

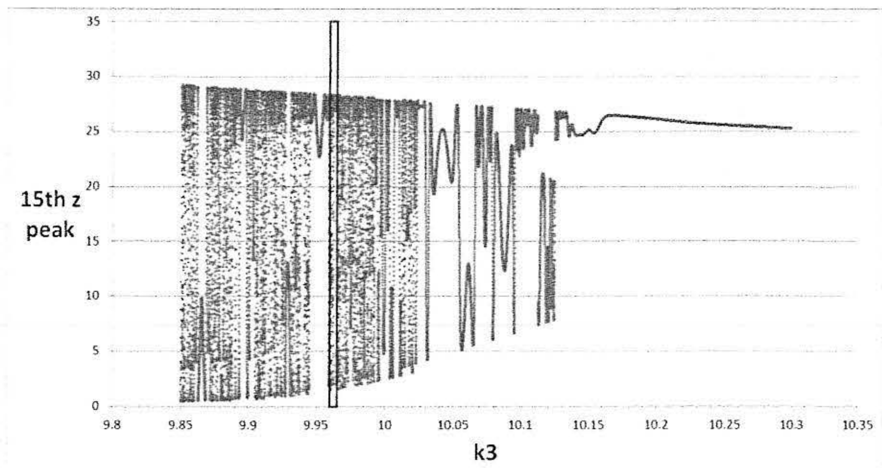

Figure 20: The 15th z peak values for each k_3 value.

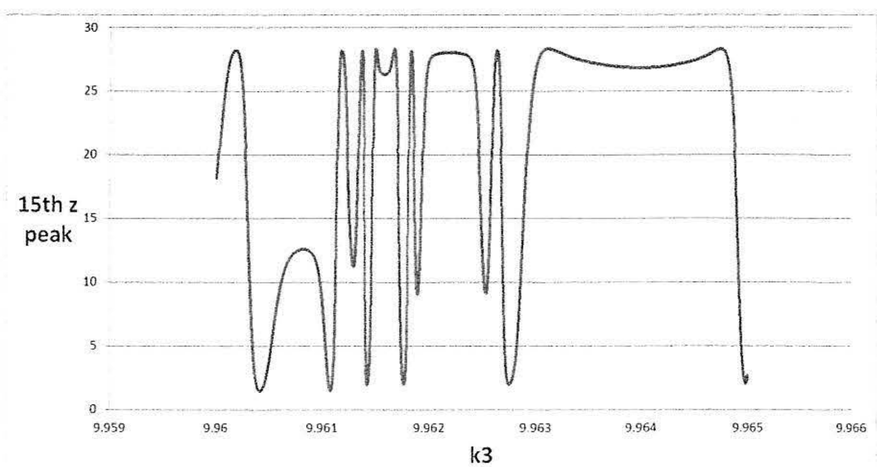

Figure 21: The 15th z peak values for each k_3 value, with higher resolution.

One possible explanation could be found by looking at how the variables are related to each other. Note that $\frac{dx}{dt}$ is dependent on all three variables, x, y, and z, whereas $\frac{dy}{dt}$ is reliant on x and y, $\frac{dz}{dt}$ on x and z. Perhaps then the discontinuities are due to x being more complicated. It is coupled to both y and z, whereas the latter two variables are only coupled to x (Scheme 4).[vi]

Investigating the 15th Valley

What about the valleys? We can construct a similar image for the 15th valley of the Williamowski-Rössler (Figure 22), and we find, unsurprisingly, that the valleys for x exhibit similar behavior to its peaks, with sections of continuity that are broken by discontinuities.

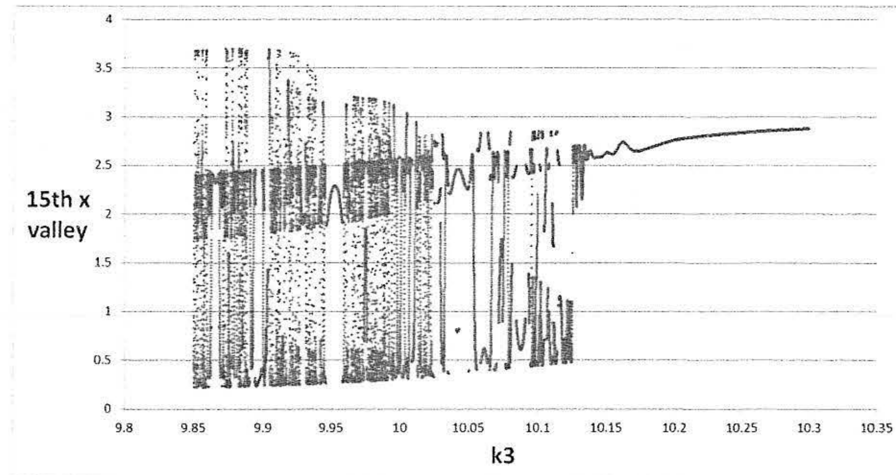

Figure 22: The 15th x valley values for each k_3 value.

However, unlike the peak graph (Figure 14), there is no discernable "line" of valley values that give an indicator of where to enforce a minimum or maximum. In the case of peak values of x, there was a discernable line of peak values that gave an indicator of where to begin in the inflation process, as this line needed to be eliminated. So if we are to force this to be continuous in a similar way as we did with the peaks,

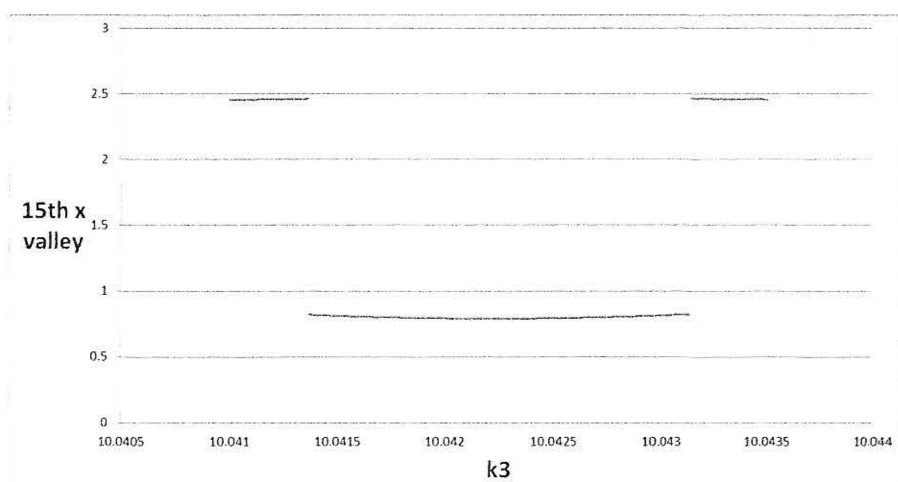

Figure 23: The 15th x valley values for each k_3 value, with higher resolution.

there is no readily available minimum or maximum to enforce.

In order to gain an appreciation of the effect enforcing a minimum ω has on the graph, we will zoom in on the graph of x's 15^{th} valley values (Figure 22), giving us a much higher resolution with which to work.

Our window will be $10.041 \leq k_3 \leq 10.0435$, which, uninflated, displays clear brokenness (Figure 23).

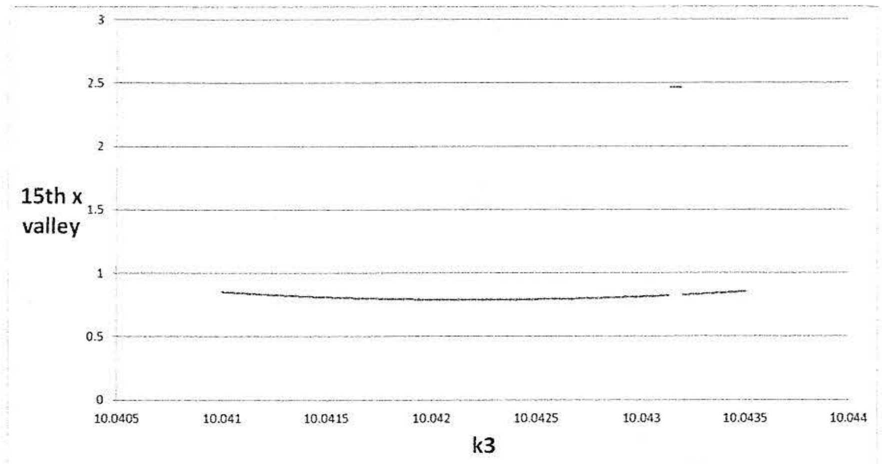

Figure 24: The 15^{th} x valley values for each k_3 value, with higher resolution, $\omega = .45$.

After several tests (data not shown), it was found that adding a minimum requirement ω for x's valley to qualify only becomes effective after we have crossed a minimum of $\omega = .45$. The leftmost discontinuity begins to disappear at $\omega \approx .4583$ (Figure 24), but reappears swiftly thereafter, while the right discontinuity begins to disappear at $\omega \approx .458835$, again reappearing quickly thereafter (Figure 23).

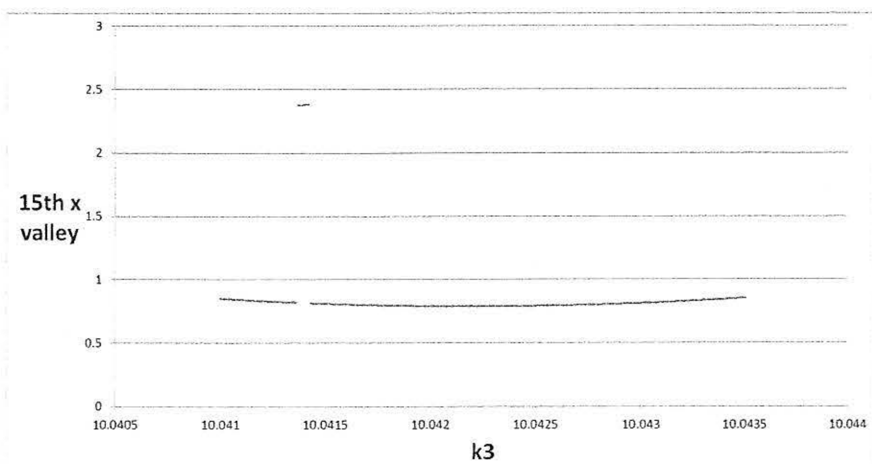

Figure 25: The 15^{th} x valley values for each k_3 value, with higher resolution, $\omega = .4583$.

However, things become even odder when the ω becomes .5 (Figure 26). The graph flips on itself, the lower edge becomes the upper, and the upper edges become lower (compare Figures 23 and 26). A look into the actual values reveals that this leap happens between approx. $10.04136 < k_3 < 10.04137$ and $10.04313 < k_3 < 10.04314$ for both inflated and uninflated systems. The leap is the same, just inversed.

So, while there has been no continuity found, we have indeed found that there are ways of manipulating the graph and forcing sections of continuity to appear. None of the minimums listed forced the whole of the graph to become continuous like our peak graph, this only serves as an example of how the graph may be changed.

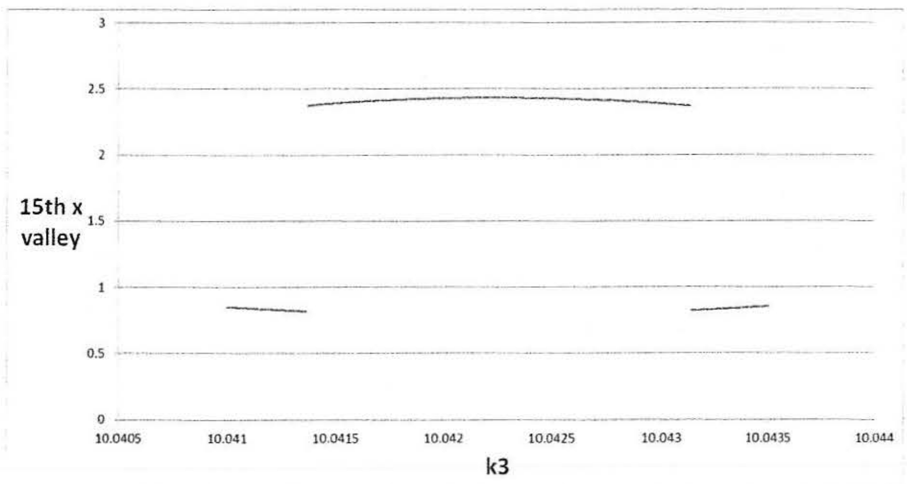

Figure 26: The 15th x valley values for each k_3 value, with higher resolution, $\omega = .5$.

Conclusions

Chaos is deterministic, not random. And so it is unsurprising that there are patterns within it. However, it is surprising where these patterns can be found, and this is certainly the case here. We have found that there is broken continuity when tracing the 15th peak of the Williamowski-Rössler Network, but that this brokenness is smoothed over when artificially raising the requirements for what can be considered a peak. Continuity follows from inflation. Similar results, though not as complete, were found with the valleys of the network as well.

This continuity is not natural, it is forced. However, continuity is no trivial thing, and a slight modification that affects every part of the graph was enough to enforce total continuity. Even with the peaks, raising the requirements for qualifying for a peak forced a section of a graph to become close to continuous, as well as exhibit very strange behavior (flipping). Peak and trough inflation therefore are worthy fields of exploration, as they have already yielded interesting and relevant results to the horizon of chaos.

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[ii] Strogatz, Steven H. Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering. Reading, MA: Addison-Wesley Pub., 1994. Print. pp. 323. Emphasis and bolding in original.

[iii] Ibid. pp. 321-322.

[iv] Bush, Aaron. *A Discovery and Analysis of the Hidden Structure in Bifurcation Diagrams of Chaotic Chemical Mechanisms*. May 6th, 2009. Spring Arbor University. Print.

[v] Ibid. pp. 355.

[vi] Dr. Kuntzelman, Personal Correspondence